COMBINATORICA 2 (1) (1982) 37—51

# CANONICAL EDGE-COLOURINGS OF LOCALLY FINITE GRAPHS

A. J. W. HILTON

*Received 6 December 1980*

A variety of results on edge-colourings are proved, the main one being the following: if $G$ is a graph without loops or multiple edges and with maximum degree $\Delta=\Delta(G)$, and if $\nu$ is a given integer $1\leqq\nu\leqq\Delta(G)$, then $G$ can be given a proper edge-colouring with the colours $c_1, \dots, c_{\Delta+1}$ with the additional property that any edge coloured $c_\mu$ with $\mu\geqq\nu$ is on a vertex which has on it edges coloured with at least $\nu-1$ of $c_1, \dots, c_\nu$.

In this note we consider graphs without loops in which multiple edges may occur. The degree of each vertex will be finite (i.e. the graphs are *locally finite*). An *edge-colouring* of a graph $G$ is an assignment of colours to the edges of $G$ such that each edge receives exactly one colour. A *proper edge-colouring* of $G$ is an edge-colouring such that no two edges on the same vertex receive the same colour. If there is a proper edge-colouring of $G$ with a finite number of colours, the *chromatic index* (edge chromatic number) $\chi'(G)$ of $G$ is defined to be the least integer $j$ for which there is a proper edge colouring of $G$ with $j$ colours.

If a finite graph is properly edge-coloured with colours $c_1, c_2, \dots$ then a "canonical" proper edge-colouring may be derived from it, if by "canonical" we mean for the moment that, for each edge $e$, if $e$ is coloured $c_i$, then each of $c_1, \dots, c_{i-1}$ occurs on an edge on one or other of the vertices of $e$. The method of derivation is almost obvious: we iterate the following procedure. Find an edge $e$ coloured $c_j$ which does not have each of $c_1, \dots, c_{j-1}$ on an edge on one or other of the vertices of $e$; then recolour $e$ with $c_i$ where $i$ is the least integer such that $c_i$ does not occur on either of the vertices of $e$. Since the graph is finite the procedure must terminate; however, the result extends to locally finite infinite graphs.

It is of interest to see to what extent a proper edge-colouring can be made even more "canonical". Here we consider proper edge-colourings in which if an edge is coloured $c_i$ then, as above, each of $c_1, \dots, c_{i-1}$ occurs on edges on one or other of the vertices of $e$, but further, as many as possible of $c_1, \dots, c_{i-1}$ occur on at least one of the vertices of $e$. It is clear that at least $\left[\frac{i}{2}\right]$ of $c_1, \dots, c_{i-1}$ must

AMS subject classification (1980): 05 C 15

occur on edges on at least one of the vertices of an edge coloured $c_i$ in a proper edge-colouring which is "canonical" in the first sense. However, we can always do better than that.

Our first result in this direction is the following.

**Theorem 1.** *A locally finite bipartite multigraph G has a proper edge-colouring with finitely many or denumerably many colours* $c_1, c_2, \ldots$ *such that for each* $i \geqq 2$, *each edge coloured* $c_i$ *is on a vertex which also has on it edges coloured* $c_1, \ldots, c_{i-1}$.

Notice that the theorem of König [5] that a bipartite graph of maximum degree $\Delta$ can be properly edge-colouréd with $\Delta$ colours, is an immediate corollary of Theorem 1.

Theorem 1 is a special case of the following more general theorem.

**Theorem 2.** *Let j be a positive integer. Let G be a locally finite multigraph which has no loops. If either*

(i) *j is even, or*

(ii) *j is odd and G contains no subgraph with an odd number of edges which is regular of degree* $2j$,

*then G can be edge-coloured with finitely many or denumerably many colours* $c_1, c_2, \ldots$ *in such a way that*

(iii) *no vertex has more than j edges of each colour on it,*

(iv) *for each* $i \geqq 2$, *each edge coloured* $c_i$ *is on a vertex which also has on it j edges coloured* $c_\nu$ *for each* $\nu \in \{1, \ldots, i-1\}$.

Given a set $E$ of edges of a graph $G$, it will be convenient to refer to the graph consisting of the edge set $E$, and those vertices of $G$ which are on edges of $E$, as the graph $E$. If $G$ has a greatest degree, it will be denoted by $\Delta(G)$.

We turn now to the proof of theorem 2. First we recall that in the case when $G$ is finite a slightly weaker version of theorem 2 was stated by Hilton in [4]. The argument is essentially that used by Geller and Hilton in [3]; however, we give it in full here for the sake of completeness.

**The proof of theorem 2.**

*Case 1. G is finite.*

Let $\Delta = \Delta(G)$ denote the greatest degree in $G$. In this case, theorem 2 was proved in [3, corollary 2.1], except that (iv) was not proved; the number of colours used was $\left[\frac{\Delta+j-1}{j}\right]$, where $[y]$ denotes the greatest integer not greater than $y$. Let $x = \left[\frac{\Delta+j-1}{j}\right]$. We prove theorem 2 in this case by induction on $x$.

If $x=1$ there is nothing to prove. Let $y>1$ and as an induction hypothesis suppose theorem 2 is true in this case with $x=y-1$. Now consider the case when $x=y$. Let $G$ be edge-coloured with $x$ colours $c_1, \ldots, c_x$ so that (iii) is satisfied. For $1 \leqq i \leqq x$, let $E_i$ be the set of edges coloured $c_i$. If an edge $e$ in $E_x$ is on two vertices both of which have degree less than $j(x-1)$ in $E_1 \cup \ldots \cup E_{x-1}$ then remove this edge from $E_x$ and add it to $E_1 \cup \ldots \cup E_{x-1}$. The graph formed from $e$ and $E_1 \cup \ldots \cup E_{x-1}$ also has maximum degree $\leqq j(x-1)$. Now continue to transfer such edges $e$ until it is no longer possible to do so. Let $E_x^*$ be the set of edges of $E_x$

which remain when this process stops. Then each edge of $E_x^*$ is on at least one vertex which has degree $j(x-1)$ in $G-E_x^*$. Therefore in any edge-colouring which satisfies (iii) and in which the edges of $E_x^*$ receive the colour $c_\Delta$, each edge coloured $c_\Delta$ is on a vertex which also has on it $j$ edges coloured $c_v$ for each $v\in\{1, ..., \Delta-1\}$. By induction $G-E_x^*$ can be edge-coloured so as to satisfy (iii) and (iv), and therefore so can $G$. Case 1 now follows by induction.

*Case 2. G is infinite.*

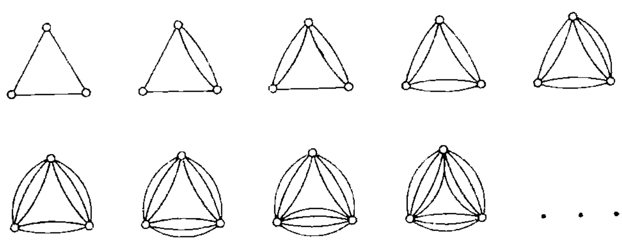
Let $C=\{c_1, c_2, ...\}$ be a set of colours, denumerable if $G$ has no greatest degree, or cardinality $\left[\frac{\Delta(G)+j-1}{j}\right]$ if $G$ has a greatest degree. Let $E$ be the set of edges of $G$. For each finite subset $E_L$ of $E$, let $f_L$: $E_L\to\{c_1, ..., c_x\}$, where $x=\left[\frac{\Delta(E_L)+j-1}{j}\right]$, be a function which assigns the colours $c_1, ..., c_x$ to the edges of $E_L$ such that the rules (iii) and (iv) are obeyed. By Rado's selection principle [6, 7] there is a function $f^*$: $E\to C$ such that, given any finite subset $E_L$ of $E$ there is a finite subset $E_M$ of $E$ containing $E_L$ with $f^*(e)=f_M(e)$ for each $e\in E_L$. Thus $f^*$ assigns the colours $\{c_1, c_2, ...\}$ to the edges $E$ so that (iii) and (iv) are satisfied, as required. ∎

Of course, the greatest interest in edge-colouring graphs lies in the case when the edge-colourings are proper. What can we say in general about canonical proper edge-colourings of locally finite multigraphs?

**Theorem 3.** *A locally finite multigraph G can be properly edge-coloured with finitely many or denumerably many colours* $c_1, c_2, \ldots$ *in such a way that, for each* $i\geqq 2$, *each edge coloured* $c_i$ *is on a vertex which also has on it at least* $\left[\frac{2(i-1)}{3}\right]$ *edges coloured from* $\{c_1, ..., c_{i-1}\}$.

**Proof.** By Theorem 2 with $j=2$, $G$ can be partitioned into subgraphs $G_1, G_2, \ldots$ each of which consists of disjoint paths and circuits, and such that each edge of $G_l$ is on a vertex which has degree 2 in each of $G_1, G_2, ..., G_{l-1}$. Now, for each $i\geqq 1$, colour the edges of $G_i$ properly with the colours $c_{3i-2}$, $c_{3i-1}$, $c_{3i}$, using the colour $c_{3i}$ only on edges which are not end edges of maximal paths of $G_i$. Then edges coloured $c_{3l-2}(c_{3l-1}, c_{3l})$ are each on a vertex which has on it edges coloured with $2(l-1)$ $(2(l-1), 2(l-1)+1$ respectively) of the colours $c_1, ..., c_{3l-2}$ $(c_1, ..., c_{3l-1}; c_1, ..., c_{3l}$ respectively). The theorem now follows. ∎

The following graph shows that the figure given in Theorem 3 is best possible.



We can similarly derive other consequences of theorem 2 when, for some odd integer $j$, $G$ does not contain a regular subgraph of degree $2j$ with an odd number of edges. The most interesting of these is the following theorem.

**Theorem 4.** *A locally finite multigraph $G$ which contains no regular subgraph of degree 6 with an odd number of edges may be properly edge-coloured with finitely many or denumerably many colours $c_1, c_2, \ldots$ in such a way that, for each $i \geqq 2$, each edge coloured $c_i$ is on a vertex which also has on it at least $3 \times \left[\frac{i-1}{4}\right]$ edges coloured from $\{c_1, \ldots, c_{i-1}\}$.*

**Proof.** By theorem 2 with $j=3$, $G$ can be partitioned into subgraphs $G_1, G_2, \ldots$ each of which has maximum degree at most 3, and such that, for each $l>1$, each edge of $G_l$ is on a vertex of degree 3 in each of $G_1, G_2, \ldots, G_{l-1}$. Each of $G_1, G_2, \ldots$ can be properly edge-coloured with $\leqq 4$ colours (see Bosák [2]). The theorem now follows. ▮

We were unable to decide whether the following conjecture concerning simple graphs was true.

**Conjecture 1.** *A locally finite simple graph $G$ can be properly edge-coloured with finitely many or denumerably many colours $c_1, c_2, \ldots$ in such a way that, for each $i \geqq 2$, each edge coloured $c_i$ is on a vertex $v$ which also has on it edges coloured $c_1, \ldots, c_{i-1}$, except for the possibility that, for one $j$, $1 \leqq j \leqq i-1$, there may be no edge coloured $c_j$ on $v$.*

However, we were able to prove the following theorem, which shows that, for any $\nu$, if the set of colours is split into two subsets $\{c_1, \ldots, c_\nu\}$, $\{c_{\nu+1}, c_{\nu+2}, \ldots\}$ then there is a proper edge-colouring in which each edge coloured $c_\mu$ with $\mu \geqq \nu+1$ is on a vertex which also has on it edges of all, or all but one, of the colours $c_1, \ldots, c_\nu$.

**Theorem 5.** *Let $G$ be a locally finite simple graph, and let $\nu \geqq 1$ be an integer such that $\nu \leqq$ maximum degree, if $G$ has one. Then $G$ can be properly edge-coloured with finitely many or denumerably many colours $c_1, c_2, \ldots$ in such a way that*

(i) *any edge coloured $c_\mu$ with $\mu \geqq \nu+1$ is on a vertex which has on it edges coloured with at least $\nu-1$ of $c_1, \ldots, c_\nu$,*

(ii) *any edge coloured $c_\nu$ is on a vertex which has on it edges coloured with at least $\nu-2$ of $c_1, \ldots, c_{\nu-1}$.*

In the statement of Theorem 5, (i) and (ii) together are equivalent to:

(iii) *any edge coloured $c_\mu$ with $\mu \geqq \nu$ is on a vertex which has on it edges coloured with at least $\nu-1$ of $c_1, \ldots, c_\nu$.*

**Proof.** *Case 1. $G$ is finite.*

First we introduce some terminology. If a vertex has on it edges of all, or all but one, of the colours $c_1, \ldots, c_\lambda$, for some $\lambda$, we shall call it a *$\lambda$-canonically coloured* vertex. If an edge is on a $\lambda$-canonically coloured vertex we shall call it a *$\lambda$-canonically coloured edge.* Clearly any edge or vertex which is $\lambda$-canonically coloured for some $\lambda>1$ is also $(\lambda-1)$-canonically coloured. A maximal path or an even circuit of edges coloured $\alpha$ or $\beta$ will be called a *maximal $(\alpha, \beta)$-chain.* If $i<j$ we shall say that the colour $c_i$ is *less than* the colour $c_j$, and write $c_i<c_j$.

Starting from a proper edge-colouring of $G$ with $c_1, \ldots, c_{\Delta+1}$ we shall recolour in several stages, keeping the edge-colouring proper at each stage. At the end of the first stage each edge coloured $c_\Delta$ or $c_{\Delta+1}$ is $\Delta$-canonically coloured. At the end of the $j$-th stage ($j \geqq 1$), each edge coloured with one of $c_{\Delta+1}, c_\Delta, \ldots, c_{\Delta+1-j}$ will be $(\Delta+1-j)$-canonically coloured.

We now suppose that $\Delta \geqq \lambda \geqq \nu$, and we describe the $(\Delta-\lambda+1)$-th stage of this recolouring process. If $\lambda = \Delta$, so that we are at the first stage of this recolouring process, then we make no assumption at all at the start of this stage except that $G$ is properly edge-coloured with $c_1, \ldots, c_{\Delta+1}$. If $\Delta > \lambda$ then at the start of this stage we assume that each edge coloured $c_{\Delta+1}, c_\Delta, \ldots, c_{\lambda+1}$ is $(\lambda+1)$-canonically coloured. Recall that this implies that each of these edges is $\lambda$-canonically coloured.

Let $P_0$ be the set of edges coloured $c_{\lambda+1}$ or $c_\lambda$ which are $\lambda$-canonically coloured. Let $R_0$ be the set of edges coloured $c_\lambda$ which are not in $P_0$, that is to say, are not $\lambda$-canonically coloured. We may assume that any edge coloured with $c_\lambda$ or $c_{\lambda+1}$ which is not adjacent to another edge coloured $c_{\lambda+1}$ or $c_\lambda$ respectively is in fact coloured $c_\lambda$. Notice that if the colours are interchanged on a maximal $(c_\lambda, c_{\lambda+1})$-path then any edge coloured $c_\lambda$ which was in $P_0$ and is not an end edge of the path remains $\lambda$-canonically coloured, but an end edge which was coloured $c_\lambda$ may not remain $\lambda$-canonically coloured after the colours are interchanged on the path.

If an edge $e \in P_0$ and the vertices of $e$ are $u$, $v$ then we shall also write $u \in P_0$, $v \in P_0$, etc.; if the two vertices of an edge $e$ are $a$ and $b$ then $e$ will also be denoted by $ab$ or $ba$. We shall call a maximal $(c_\lambda, c_{\lambda+1})$-path trivial if it consists of only one edge.

If we are not at the final stage, and if $R_0 = \emptyset$ then each edge coloured $c_{\Delta+1}, \ldots, c_\lambda$ is $\lambda$-canonically coloured and so we may pass on to the next stage, i.e. the $(\Delta-\lambda+2)$-th stage of this recolouring process. If $R_0 = \emptyset$ and we are at the last stage, i.e. the $(\Delta-\nu+1)$-th stage, then the conditions of the theorem are satisfied. Therefore suppose that $R_0 \neq \emptyset$. Then each edge coloured $c_{\Delta+1}, c_\Delta, \ldots, c_{\lambda+1}$ is $\lambda$-canonically coloured, but there is an edge coloured $c_\lambda$ which is not $\lambda$-canonically coloured. We shall recolour some of the edges. Let $P_1$, $R_1$ be the corresponding sets of edges after the recolouring. Iterating our recolouring process will give a finite sequence $(P_0, R_0)$, $(P_1, R_1)$, $\ldots$, $(P_l, R_l)$, in which

$$|R_i| \geqq |R_{i+1}| \qquad (1 \leqq i \leqq l-1),$$

$$|R_i| = |R_{i+1}| \Rightarrow |R_{i+1}| > |R_{i+2}| \quad (1 \leqq i \leqq l-2),$$

and

$$R_l = \emptyset .$$

In each case where $|R_i| = |R_{i+1}|$ no end edge of a maximal $(c_\lambda, c_{\lambda+1})$-chain is in $R_i$, but in such a case the process we describe rearranges the edge-colouring so that at least one such end edge is in $R_{i+1}$. For each $i$ it remains true that each edge coloured $c_{\Delta+1}, c_\Delta, \ldots, c_{\lambda+1}$ is $\lambda$-canonically coloured.

Thus suppose $e \in R_0$. Then $e$ is coloured $c_\lambda$. We may assume that either $e$ is an end edge of the (possibly trivial) maximal $(c_\lambda, c_{\lambda+1})$-chain on which it lies, or else that all end edges of all maximal $(c_\lambda, c_{\lambda+1})$-chains are in $P_0$. We may further assume that, if $e$ is not an end edge, then all edges between $e$ and one of the end edges, if there are any, are in $P_0$. A final assumption is that $e$ may only be in a

$(c_\lambda, c_{\lambda+1})$-circuit when all edges of all (possibly trivial) maximal $(c_\lambda, c_{\lambda+1})$-paths are in $P_0$.

Let $e = ab_1$, where $a, b_1 \in V(G)$; if $e$ is the end edge of a (possibly trivial) maximal $(c_\lambda, c_{\lambda+1})$-chain, let $a$ be an end vertex of this path; if $e$ is not the end edge and the maximal $(c_\lambda, c_{\lambda+1})$-chain is a path, let all edges between $a$ and the end on the opposite side of $a$ to $b_1$ be in $P_0$.

Now remove the edge $e$ (but not the vertices $a$, $b_1$) from $G$.

Let $\alpha$ be the least colour missing from $a$, and let $\beta_1$ be the least colour missing from $b_1$. Clearly $\alpha$ and $\beta_1$ are both $\leqq c_{\lambda-2}$. If $\alpha$ is missing from $b_1$ then we can insert the edge $e$ recoloured $\alpha$. If this should produce an edge $e^*$ coloured $c_{\lambda+1}$ which is not adjacent to an edge coloured $c_\lambda$, then change the colour on $e^*$ to $c_\lambda$; it is then in $P_1$. Then clearly $P_1$ and $R_1$ satisfy Chart 1.

| $P_1 = P_0$ | $R_1 = R_0 \setminus \{e\}$ |
|---|---|

*Chart 1*

We deal similarly with the case when $\beta_1$ is missing from $a$, and again $P_1$ and $R_1$ satisfy Chart 1. Therefore suppose from now that $\alpha$ is present at $b_1$, $\beta_1$ is present at $a$. Then $\alpha \neq \beta_1$. Let the edge on $a$ coloured $\beta_1$ be called $ab_2$.

We now iterate a certain procedure. We describe the $i$-th stage of it, where $i \geqq 2$. Suppose that, prior to the $i$-th stage, certain neighbours of $a$ have been labelled $b_1, b_2, \dots, b_i$, that $ab_2, ab_3, \dots, ab_i$ are coloured $\beta_1, \dots, \beta_{i-1}$ respectively, that $\beta_1, \dots, \beta_{i-1}$ are distinct and not equal to $\alpha$ or $c_\lambda$, that $\beta_1, \dots, \beta_{i-1}$ are the least colours missing from $b_1, \dots, b_{i-1}$ respectively, and that $\alpha$ is present at $b_1, \dots, b_{i-1}$. This is illustrated in part by Figure 1.

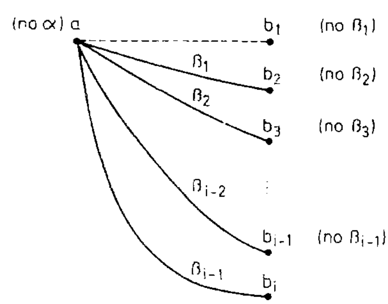



*Fig. 1*

Let $\beta_i$ be the least colour missing from $b_i$. We consider various cases.

*Case 1a.* $\beta_i = c_\lambda$. Then recolour $ab_i$ with $c_\lambda$, $ab_{i-1}$ with $\beta_{i-1}, \dots, ab_2$ with $\beta_2$ and insert $e$ recoloured $\beta_1$.

Consider edges coloured $c_\mu$, where $\mu \geqq \lambda + 1$. We assume that each edge coloured $c_\mu$ before this recolouring was $\lambda$-canonically coloured. After the recolouring the colours present at each vertex other than $a, b_1, \dots, b_i$ remain unaltered. At each of $b_1, \dots, b_i$ a colour is removed, but it is replaced by the least colour which had previously been absent. At $a$ the set of colours present remains unaltered. Thus, any edge which is coloured $c_\mu$ both before and after this recolouring remains $\lambda$-canonically coloured. If an edge $ab_\sigma$, where $1 \leqq \sigma \leqq i$, acquires the colour $c_\mu$ during this recolouring, then $c_\mu$ was previously the least colour absent from $b_\sigma$, so after the recolouring $b_\sigma$ has on it all, or all but one, of $c_1, \ldots, c_{\mu-1}$, and so $ab_\sigma$ is $(\mu-1)$-canonically coloured, and therefore $\lambda$-canonically coloured. Thus any edge coloured $c_\mu$ after this recolouring is $\lambda$-canonically coloured.

It remains to investigate $P_1$ and $R_1$ in this case.

*Case 1a* (i). $c_{\lambda+1} \notin \{\beta_1, \ldots, \beta_{i-1}\}$. Then $P_1$ and $R_1$ satisfy Chart 2.

| | |
|---|---|
| $P_1 = P_0 \cup \{ab_i\}$ | $R_1 = R_0 \setminus \{e\}$ |

*Chart 2*

*Case 1a* (ii). $c_{\lambda+1} = \beta_{i-1}$. The "before" and "after" colours of $ab_{i-1}$ and $ab_i$ are given in Figure 2.

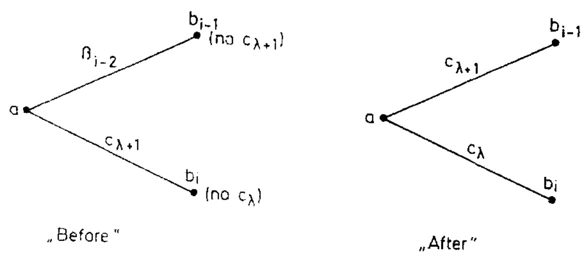



*Fig. 2*

Then $P_1$ and $R_1$ again satisfy Chart 2.

*Case 1a* (iii). $c_{\lambda+1} \in \{\beta_2, \ldots, \beta_{i-2}\}$. Let $c_{\lambda+1} = \beta_\sigma$ $(2 \leqq \sigma \leqq i-2)$. Then the "before" and "after" colours of $ab_\sigma$, $ab_{\sigma+1}$ and $ab_i$ are given in Figure 3.

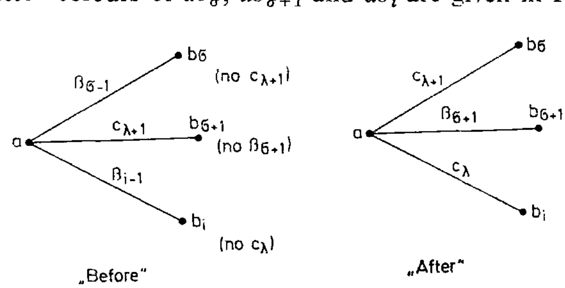



*Fig. 3*

Then $ab_{\sigma+1} \in P_0$ and $ab_\sigma, ab_i \in P_1$. Therefore $P_1$ and $R_1$ satisfy Chart 3.

| | |
|---|---|
| $P_1 = (P_0 \cup \{ab_\sigma\} \cup \{ab_i\}) \setminus \{ab_{\sigma+1}\}$ | $R_1 = R_0 \setminus \{e\}$ |

*Chart 3*

*Case 1a* (iv). $c_{\lambda+1}=\beta_1$. This does not occur, for otherwise $b_1$, and therefore $e$, would be $\lambda$-canonically coloured.

*Case 1b. $\beta_i \neq c_\lambda$.*

*Case 1b* (i). *$\alpha$ is not on any edge at $b_i$.* In this case we recolour the edge $ab_i$ with $\alpha$, recolour $ab_{i-1}$ with $\beta_{i-1}$, $ab_{i-2}$ with $\beta_{i-2}, \dots, ab_2$ with $\beta_2$ and reinsert $e$ recoloured $\beta_i$.

As in Case 1a, an edge coloured $c_\mu$ with $\mu \geqq \lambda+1$ after this recolouring is $\lambda$-canonically coloured.

*Case 1b* (i) 1. $c_{\lambda+1} \notin \{\beta_1, \dots, \beta_{i-1}\}$. Then $P_1$ and $R_1$ again satisfy Chart 1.

*Case 1b* (i) 2. $c_{\lambda+1}=\beta_\sigma$ *for some* $\sigma \in \{2, \dots, i-1\}$. The "before" and "after" colours of $ab_\sigma$ and $ab_{\sigma+1}$ are given in Figure 4.

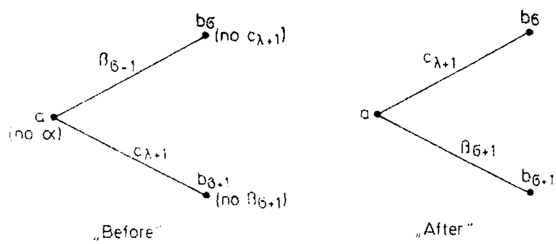



*Fig. 4*

Clearly $ab_{\sigma+1} \in P_0$ and $ab_\sigma \in P_1$. Then $P_1$ and $R_1$ again satisfy Chart 3.

*Case 1b* (i) 3. $c_{\lambda+1}=\beta_1$. This case does not occur for the same reason as Case 1a (iv).

*Case 1b* (ii). *$\alpha$ is on an edge at $b_i$.*

*Case 1b* (ii) 1. *$\beta_i$ is not on an edge on $a$.* Then we recolour the edge $ab_i$ with $\beta_i$, $ab_{i-1}$ with $\beta_{i-1}, \dots, ab_2$ with $\beta_2$ and reinsert the edge $e$ recoloured $\beta_1$.

As in Case 1a, any edge coloured $c_\mu$ with $\mu \geqq \lambda+1$ after this recolouring is $\lambda$-canonically coloured.

*Case 1b* (ii) *1a.* $c_{\lambda+1} \notin \{\beta_1, \dots, \beta_i\}$. Then $P_1$ and $R_1$ again satisfy Chart 1.

*Case 1b* (ii) *1b.* $c_{\lambda+1}=\beta_\sigma$ *for some* $\sigma \in \{2, \dots, i-1\}$. Then $P_1$ and $R_1$ again satisfy Chart 3, for the same reasons as in Case 1b (i) 2.

*Case 1b* (ii) *1c.* $c_{\lambda+1}=\beta_1$. This case does not occur for the same reason as Case 1a (iv).

*Case 1b* (ii) *1d.* $c_{\lambda+1}=\beta_i$. Then $ab_i \in P_1$ and $P_1$ and $R_1$ again satisfy Chart 2.

*Case 1b* (ii) *2. $\beta_i$ is on an edge on a.*

*Case 1b* (ii) *2a.* $\beta_i \notin \{\beta_1, \dots, \beta_{i-1}\}$. Then denote the edge on $a$ coloured $\beta_i$ by $ab_{i+1}$ and proceed to the $(i+1)$-th step of this recolouring process.

*Case 1b* (ii) *2b.* $\beta_i=\beta_j$ *for some* $j \in \{1, \dots, i-1\}$.

*Case 1b* (ii) *2b* (i). $\beta_i \neq c_{\lambda+1}$. The situation is illustrated in Figure 5.

*Case 1b* (ii) *2b* (i) *1.* $\beta_i < c_\lambda$. Let $\Pi$ denote the maximal $(\alpha, \beta_i)$-path starting at $b_i$, and let $x$ denote the vertex at the other end of $\Pi$.

*Case 1b* (ii) *2b* (i) *1a.* $x=b_\tau$ *for some* $\tau \in \{2, \dots, i-1\}$. Interchange the colours on the edges of $\Pi$, recolour $ab_\tau$ with $\alpha$, $ab_{\tau-1}$ with $\beta_{\tau-1}, \dots, ab_2$ with $\beta_2$ and reinsert $e$ coloured $\beta_1$. As in Case 1a, any edge coloured $c_\mu$ with $\mu \geqq \lambda+1$ after this recolouring is $\lambda$-canonically coloured.

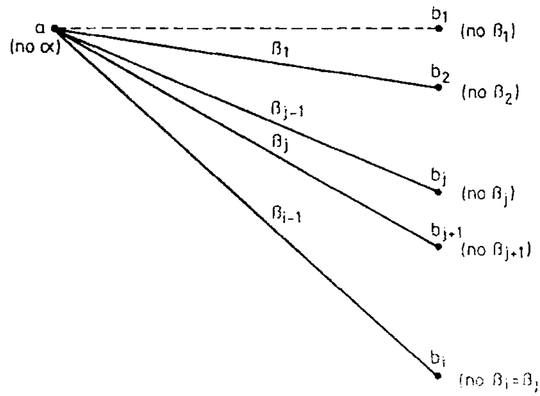



*Fig. 5*

*Case 1b* (ii) *2b* (i) *1a* (i). $c_{\lambda+1} \notin \{\beta_1, \ldots, \beta_{\tau-1}\}$. Then $P_1$ and $R_1$ again satisfy Chart 1.

*Case 1b* (ii) *2b* (i) *1a* (ii). $c_{\lambda+1}=\beta_\sigma$ *for some* $\sigma \in \{2, \ldots, \tau-1\}$. Then $P_1$ and $R_1$ again satisfy Chart 3, the reasons being the same as in Case 1b (i) 2.

*Case 1b* (ii) *2b* (i) *1a* (iii). $c_{\lambda+1}=\beta_1$. This case does not arise for the same reason as Case 1a (iv).

*Case 1b* (ii) *2b* (i) *1b*. $x=a$. Then $\Pi$ passes through $b_{j+1}$. Interchange the colours on the edges of $\Pi$, recolour $ab_j$ with $\beta_j$, $ab_{j-1}$ with $\beta_{j-1}, \ldots, ab_2$ with $\beta_2$ and insert $e$ coloured $\beta_1$.

As in Case 1a, any edge coloured $c_\mu$ with $\mu \geqq \lambda+1$ after this recolouring is $\lambda$-canonically coloured.

The three cases $c_{\lambda+1} \notin \{\beta_1, \ldots, \beta_{j-1}\}$, $c_{\lambda+1}=\beta_\sigma$ for some $\sigma \in \{2, \ldots, j-1\}$ and $c_{\lambda+1}=\beta_1$ are essentially the same as the corresponding subcases of Case 1b (ii) 2b (i) 1a. The case $c_{\lambda+1}=\beta_j$ is excluded as we are in a subcase of Case 1b (ii) 2b (i).

*Case 1b* (ii) *2b* (i) *1c*. $x=b_1$. Interchange the colours on $\Pi$ and insert $e$ recoloured $\alpha$. Then $P_1$ and $R_1$ again satisfy Chart 1.

*Case 1b* (ii) *2b* (i) *1d*. $x \notin \{a, b_1, b_2, \ldots, b_{i-1}\}$. Interchange the colours on the edges of $\Pi$, recolour $ab_i$ with $\alpha$, $ab_{i-1}$ with $\beta_{i-1}$, $ab_{i-2}$ with $\beta_{i-2}, \ldots, ab_2$ with $\beta_2$ and reinsert $e$ coloured $\beta_1$.

If $x$ was $\lambda$-canonically coloured then it remains so, since it has one colour less than $c_\lambda$ removed from it, which is then replaced by another. Apart from this detail, the argument for this case is essentially the same as that of Case 1b (i).

*Case 1b* (ii) *2b* (i) *2*. $\beta_i > c_{\lambda+1}$. Let $\Pi_i$ ($\Pi_j$) denote the maximal $(c_{\lambda+1}, \beta_i)$-path starting at $b_i$ ($b_j$), and let $x_i$ ($x_j$) denote the vertex at the other end of $\Pi_i$ ($\Pi_j$). Note that the fact that $\beta_i > c_{\lambda+1}$ implies that there is an edge coloured $c_{\lambda+1}$ on $b_i$. Also note that if we interchange the colours on $\Pi_i$ then each edge of $\Pi_i$ continues to be $\lambda$-canonically coloured, all edges $c_\mu$ with $\mu \geqq \lambda+1$ continue to be $\lambda$-canonically coloured and all edges coloured $c_\lambda$ which are $(\lambda-1)$-canonically coloured continue to be $(\lambda-1)$-canonically coloured. Similarly with $\Pi_j$. Note also that $j \neq 1$ since $\beta_1 < c_\lambda$.

We now describe a process in which we cycle back to the start of Case 1b (ii). This cycle may be repeated several times, but it can be seen that the number of such cycles is finite. For if we write

$$(\beta_1, \beta_2, \ldots, \beta_i) \prec (\beta_1', \beta_2', \ldots, \beta_{i'}'),$$

when *either* $\beta_1=\beta_1', \ldots, \beta_{\tau-1}=\beta_{\tau-1}'$ and $\beta_\tau<\beta_\tau'$ for some $\tau$, $1\leqq\tau\leqq\max(i,i')$, *or* $i'>i$ and $\beta_1=\beta_1', \ldots, \beta_i=\beta_i'$, then we obtain a lexicographic total order on the sequences $(\beta_1, \ldots, \beta_i)$: inspecting each subcase of Case 1b (ii) 2b (i) 2 below will confirm that at the end of each cycle we obtain a sequence which is lower in the order than the sequence at the start of the cycle.

*Case 1b* (ii) *2b* (i) *2a.* $x_i\notin\{a, b_1, \ldots, b_{i-1}\}$. Interchange the colours on $\Pi_i$. Now start the argument for Case 1b (ii) again with $\beta_i=c_{\lambda+1}$.

*Case 1b* (ii) *2b* (i) *2b.* $x_i=b_j$. Then $\Pi_i=\Pi_j$. Interchange the colours on $\Pi_i$. Then the least colour missing from $b_j$ is now $c_{\lambda+1}$. Now start the argument for Case 1b (ii) again with $i$ replaced by $j$ and with $\beta_j=c_{\lambda+1}$.

*Case 1b* (ii) *2b* (i) *2c. The last edge of $\Pi_i$ is not $ab_\sigma$ for all*

$$\sigma\in\{1, \ldots, j-1, j+1, \ldots, i-1\}.$$

*Case 1b* (ii) *2b* (i) *2c* (i). $x_i=a$. Interchange the colours on $\Pi_i$. Then repeat the argument for Case 1b (ii) with $\beta_i=c_{\lambda+1}$.

*Case 1b* (ii) *2b* (i) *2c* (ii). $x_i=b_\sigma$ *for some* $\sigma\in\{1, \ldots, j-1, j+1, \ldots, i-1\}$. Interchange the colours on $\Pi_i$.

*Case 1b* (ii) *2b* (i) *2c* (ii) *1. The least colour missing at $b_\sigma$ is still $\beta_\sigma$.* Repeat the argument for Case 1b (ii) with $\beta_i=c_{\lambda+1}$.

*Case 1b* (ii) *2b* (i) *2c* (ii) *2. The least colour missing at $b_\sigma$ is no longer $\beta_\sigma$.* Then this least colour will be $c_{\lambda+1}$. Repeat the argument for Case 1b (ii) with $i$ replaced by $\sigma$ and with $\beta_\sigma=c_{\lambda+1}$.

*Case 1b* (ii) *2b* (i) *2d. The last edge of $\Pi_i$ is $ab_\sigma$ for some*

$$\sigma\in\{1, \ldots, j-1, j+1, \ldots, i-1\}.$$

*Case 1b* (ii) *2b* (i) *2d* (i). *The last edge of $\Pi_j$ is not $ab_\tau$ for some*

$$\tau\in\{1, \ldots, j-1, j+1, \ldots, i-1\}.$$

*Case 1b* (ii) *2b* (i) *2d* (i) *1.* $x_j=a$. This is impossible.

*Case 1b* (ii) *2b* (i) *2d* (i) *2.* $x_j=b_\mu$ *for some* $\mu\in\{1, \ldots, j-1, j+1, \ldots, i-1\}$, $\mu\neq\sigma$. Interchange the colours on $\Pi_j$.

*Case 1b* (ii) *2b* (i) *2d* (i) *2a. Either $\mu<j$ and the least colour missing at $b_\mu$ is still $\beta_\mu$, or $\mu>j$.* Repeat the argument for Case 1b (ii) with $i$ replaced by $j$ and with $\beta_j=c_{\lambda+1}$.

*Case 1b* (ii) *2b* (i) *2d* (i) *2b. $\mu<j$ and the least colour missing from $b_\mu$ is no longer $\beta_\mu$.* Then the least colour missing from $b_\mu$ is $c_{\lambda+1}$. Repeat the argument for Case 1b (ii) with $i$ replaced by $\mu$ and with $\beta_\mu$ replaced by $c_{\lambda+1}$.

*Case 1b* (ii) *2b* (i) *2d* (i) *3.* $x_j=b_\sigma$. This is impossible.

*Case 1b* (ii) *2b* (i) *2d* (ii). *The last edge of $\Pi_j$ is $ab_\tau$ for some*

$$\tau\in\{1, \ldots, j-1, j+1, \ldots, i-1\}.$$

This is impossible.

The next case, Case 1b (ii) 2b (ii), where $\beta_i = c_{\lambda+1}$, is rather elaborate. We shall denote it by Case I.

*Case I* $\big(=$*Case 1b* (ii) *2b* (ii)$\big)$. $\beta_i = c_{\lambda+1}$. Let $\Pi_i^*$ $(\Pi_j^*, \Pi_{j+1}^*)$ be the maximal $(c_\lambda, c_{\lambda+1})$-path whose first edge is $ab_i$ $(ab_j, ab_{j+1}$ respectively). If we interchange the colours on any one of these maximal paths, then each edge which is not an end edge and which was $\lambda$-canonically coloured remains so. An end edge coloured $c_\lambda$ such that the penultimate vertex of the path is not $\lambda$-canonically coloured, whereas the ultimate vertex is $\lambda$-canonically coloured with precisely $\lambda-1$ of $c_1, \dots, c_\lambda$ occurring on edges incident with it, will cease to be $\lambda$-canonically coloured. Let $x_i^*$ $(x_j^*, x_{j+1}^*)$ be the vertex at the other end of $\Pi_i^*$ $(\Pi_j^*, \Pi_{j+1}^*$ respectively).

*Case II.* $x_{j+1}^* \notin \{b_1, b_j\}$. Then $\Pi_j^* \neq \Pi_{j+1}^*$ so $x_j^* \neq a$. Interchange the colours on $\Pi_j^*$, recolour $ab_j$ with $c_\lambda$, $ab_{j-1}$ with $\beta_{j-1}$, $ab_{j-2}$ with $\beta_{j-2}$, ..., $ab_2$ with $\beta_2$ and reinsert $e$ recoloured $\beta_1$. Then interchange the colours on the maximal $(c_\lambda, c_{\lambda+1})$-path containing $\Pi_j^*$.

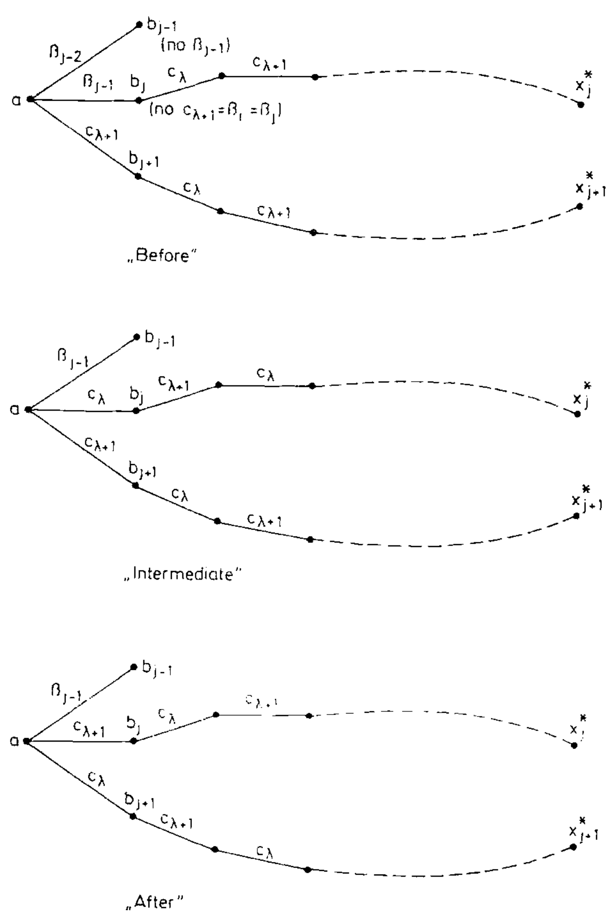



*Fig. 6*

The "before", "intermediate" and "after" colours at $ab_{j-1}$, $ab_j$ and $ab_{j+1}$ and on the paths $\Pi_j^*$ and $\Pi_{j+1}^*$ are illustrated in Figure 6.

From the way that the edge $e$ was selected it follows that all edges of $\Pi_{j+1}^*$ were in $P_0$. After the interchanges it may happen that the last edge (say $e^*$) of $\Pi_{j+1}^*$ is coloured $c_{\lambda+1}$ and is not $\lambda$-canonically coloured. We deal with this eventuality in Case I* below. But, by an argument similar to that of Case 1a, any other edge coloured $c_\mu$ with $\mu \geqq \lambda+1$ after the recolouring is $\lambda$-canonically coloured. The sets $P_1$ and $R_1$ satisfy Chart 4.

| | | |
|---|---|---|
| $e^*$ is $\lambda$-canonically coloured | $P_1 = P_0 \cup \{ab_j\}$ | $R_1 = R_0 \setminus \{e\}$ |
| $e^*$ is not $\lambda$-canonically coloured | $P_1 = (P_0 \cup \{ab_j\}) \setminus \{e^*\}$ | $R_1 = (R_0 \setminus \{e\}) \cup \{e^*\}$ |

*Chart 4*

*Case I2.* $x_{j+1}^* = b_1$. Then again $x_j^* \neq a$. In this case $e$ was part of a $(c_\lambda, c_{\lambda+1})$-circuit, so by assumption each edge of each maximal $(c_\lambda, c_{\lambda+1})$-path was in $P_0$. In particular all edges of $\Pi_j^*$ are in $P_0$. Interchange the colours on $\Pi_j^*$, recolour $ab_j$ with $c_\lambda$, $ab_{j-1}$ with $\beta_{j-1}$, $ab_{j-2}$ with $\beta_{j-2}$, ..., $ab_2$ with $\beta_2$ and reinsert $e$ recoloured $\beta_1$.

The "before" and "after" colours are the "before" and "intermediate" colours of Figure 6.

In a similar way to the previous case, the last edge of $\Pi_j^*$ may be coloured $c_{\lambda+1}$ yet not be $\lambda$-canonically coloured. Again we deal with this possibility in Case I* below. Again, any other edge coloured $c_\mu$ with $\mu \geqq \lambda+1$ after the recolouring will continue to be $\lambda$-canonically coloured. The sets $P_1$ and $R_1$ are again given by Chart 4.

*Case I3.* $x_{j+1}^* = b_j$. In this case $\Pi_j^* = \Pi_{j+1}^*$ and $x_j^* = a$. Then interchange the colours on $\Pi_i^*$, recolour $ab_i$ with $c_\lambda$, $ab_{i-1}$ with $\beta_{i-1}$, $ab_{i-2}$ with $\beta_{i-2}$, ..., $ab_2$ with $\beta_2$ and reinsert $e$ recoloured $\beta_1$. Then $ab_j$ is recoloured $c_{\lambda+1}$. Finally interchange the colours on the maximal $(c_\lambda, c_{\lambda+1})$-path containing $\Pi_j^*$. The "before", "intermediate" and "after" colours are illustrated in Figure 7.

As in the previous two cases, this process may produce an edge $e^*$ coloured $c_{\lambda+1}$ which is not $\lambda$-canonically coloured. Again this possibility is dealt with in Case I* below. This time $e^*$ will be the edge of the maximal $(c_\lambda, c_{\lambda+1})$-path we have produced which is on the vertex $b_{j+1}$. Apart from $e^*$, any other edge coloured $c_\mu$ with $\mu \geqq \lambda+1$ after this recolouring is $\lambda$-canonically coloured. The sets $P_1$ and $R_1$ satisfy Chart 5.

| | |
|---|---|
| $e^*$ is $\lambda$-canonically coloured | $P_1 = (P_0 \cup \{ab_j\} \cup \{ab_i\}) \setminus \{ab_{j+1}\}$<br>$R_1 = R_0 \setminus \{e\}$ |
| $e^*$ is not $\lambda$-canonically coloured | $P_1 = (P_0 \cup \{ab_j\} \cup \{ab_i\}) \setminus (\{ab_{j+1}\} \cup \{e^*\})$<br>$R_1 = (R_0 \setminus \{e\}) \cup \{e^*\}$ |

*Chart 5*

*Case I**. In this sub-case of Case I an edge $e^*$ has been produced being the end vertex of a maximal $(c_\lambda, c_{\lambda+1})$-path, and coloured $c_{\lambda+1}$, yet not being $\lambda$-canonically

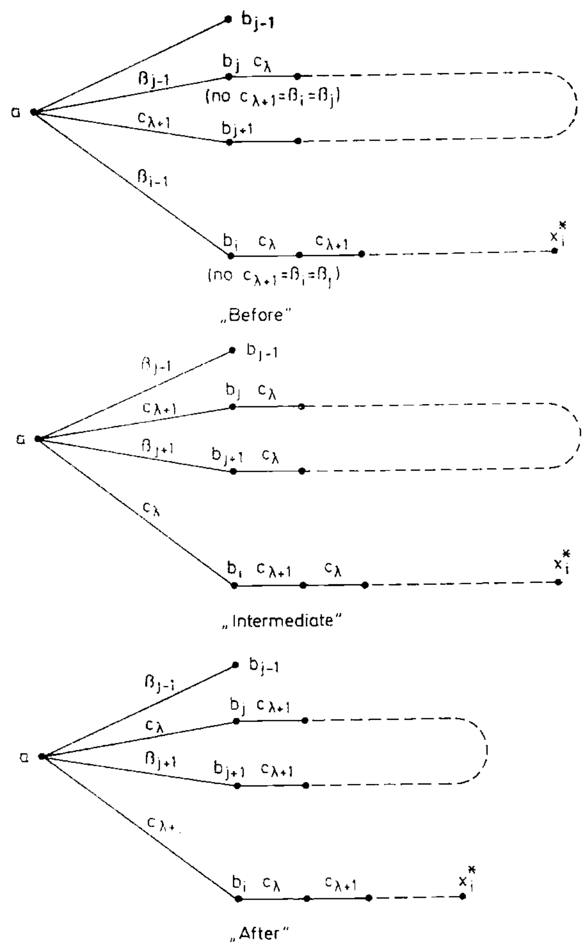



*Fig. 7*

coloured. We describe as briefly as possible a recolouring process for $e^*$ which is similar to that by which $e$ was recoloured.

We take $P_1$ and $R_1$ as they have been produced in Case I and modify them, yielding $P_1^*$ and $R_1^*$; these will then be the next 'input' in our recolouring. Let $e^*=a^*b_1^*$, where $a^*$ is the end vertex of the maximal $(c_\lambda, c_{\lambda+1})$-chain of which $e^*$ is a part. Remove the edge $e^*$. Let $\alpha^*$ be the least colour missing from $a$, and let $\beta_1^*$ be the least colour missing from $b_1^*$. Clearly $\alpha^*$ and $\beta^*$ are both $\leqq c_{\lambda-1}$. The account now proceeds as in Case 1, except that much of the description can be omitted as being inapplicable; we replace $a$ by $a^*$, $\alpha$ by $\alpha^*$, $b_1, \dots, b_i$ by $b_1^*, \dots, b_i^*$ and $\beta_1, \dots, \beta_i$ by $\beta_1^*, \dots, \beta_i^*$. The cases which may arise are (using the numbering of Case 1, but with 1 replaced by I*) Case I*a (i), Case I*b (i) 1, Case I*b (ii) 1a, Case I*b (ii) 1d, Case I*b (ii) 2a, Case I*b (ii) 2b (i) 1a (i) (the case $c_{\lambda+1} \notin \{\beta_1, \dots \dots, \beta_{j-1}\}$), Case I*b (ii) 2b (i) 1c, Case I*b (ii) 2b (i) 1d, Case I*b (ii) 2b (i) 2 (all subcases, and we cycle back to the start of Case I*); the important point to notice is that Case I*b (ii) 2b (ii) cannot arise, and so the procedure will terminate.

*Case 2. G is infinite.* This case follows from Case 1 and Rado's selection principle (as with Case 2 of Theorem 2). ∎

We next consider edge-colourings of finite Eulerian graphs (graphs all of whose degrees are even). If $G$ is an Eulerian graph, let $\chi_c(G)$ be the least number $j$ such that $E(G)$ can be coloured with $j$ colours in such a way that each colour class is the vertex-disjoint union of circuits. Call $\chi_c(G)$ the *circuit chromatic index* of $G$; by Euler's theorem $G$ is the union of edge-disjoint circuits so $\chi_c(G)$ exist. Clearly $\chi_c(G) \geqq \frac{1}{2}\Delta(G)$; Andersen showed the author that there is equality here.

**Theorem 6.** *If $G$ is a finite Eulerian graph then $\chi_c(G) = \frac{1}{2}\Delta(G)$.*

**Proof.** Let $G^*$ be the graph formed from $G$ by adjoining $\frac{1}{2}(\Delta(G) - \deg_G(v))$ loops to each vertex $v$. Then, with each loop counting 2 towards the degree of the vertex it is on, $G^*$ is regular of degree $\Delta(G)$. Therefore $G^*$ is the union of $\frac{1}{2}\Delta(G)$ edge-disjoint 2-factors. Therefore $G$ is the union of $\frac{1}{2}\Delta(G)$ colour classes, each being the union of vertex-disjoint circuits. ∎

We have, furthermore, the following analogue of Theorem 1.

**Theorem 7.** *If $G$ is a finite Eulerian graph then $E(G)$ may be coloured with colours $C_1, \dots, C_{\frac{1}{2}\Delta(G)}$ in such a way that*

(i) *each colour class is the vertex disjoint union of circuits,*

(ii) *each circuit of colour $C_j$ is on a vertex which also has on it circuits of colours $C_1, \dots, C_{j-1}$.*

**Proof.** The proof is analogous to that of Theorem 1. ∎

As an analogue of a concept of de Werra [11], for Eulerian graphs we may define an *evenly-equitable* edge colouring. This is a decomposition of $E(G)$ into colour classes $C_1, \dots, C_k$ such that

(i) $|C_i(v)|$ is even for each $v \in V(G)$ and each $i$, $1 \leqq i \leqq k$,

(ii) $|C_i(v) - C_j(v)| = 0$ or 2 for each $v \in V(G)$ and each $i, j$ with $1 \leqq i < j \leqq k$, where $C_i(v)$ denotes the number of edges of colour $C_i$ on the vertex $v$.

**Theorem 8.** *For each $k \geqq 1$, each finite Eulerian graph has an evenly equitable edge-colouring with $k$ colours.*

**Proof.** Let $G$ be edge-coloured with $C_1, \dots, C_k$ in such a way that, for each $i$, $1 \leqq i \leqq k$, the graph whose vertices are the vertices of $V(G)$ and whose edges are the edges of $C_i$ is Eulerian (it is easy to see that such an edge-colouring exists).

We now describe an algorithm which changes this edge-colouring to one of the desired type. Suppose there is a vertex $v_0$ and two colours $C_\alpha$, $C_\beta$ such that $|C_\alpha(v_0) - C_\beta(v_0)| > 2$. Let $G^*$ be the subgraph whose vertices are the vertices of $G$ and whose edges are the edges of $C_\alpha \cup C_\beta$. Let $G^{**}$ be the graph formed from $G^*$ by adding a loop at each vertex whose degree in $G^*$ is congruent to 2 (mod 4). Then each vertex of $G^{**}$ has degree congruent to 0 (mod 4) (counting 2 for the contribution from each loop). Now colour the edges of an Eulerian circuit of $G^{**}$ alternately $C_\alpha$, $C_\beta$; we see that there is an equal number of edges at each vertex of both colours (counting a loop as two edges). The colouring restricted to $G^*$ then has

$$|C_\alpha(v)-C_\beta(v)| = 0 \text{ or } 2 \quad (\forall v \in V(G)).$$

Repetition of this process clearly yields an evenly equitable edge-colouring of $G$. ▮

We have just learned that Theorems 6, 7 and 8 can all be derived from a more general result (Theorem 2.4) in [12].

Now let the *path-chromatic* index $\chi_p(G)$ of a simple graph $G$ be defined as the least number $j$ such that $E(G)$ can be coloured with $j$ colours in such a way that each colour class is the vertex-disjoint union of paths (a circuit not being counted as a path). We make the following conjecture regarding $\chi_p(G)$.

**Conjecture 2.** *Let $G$ be a simple graph. Then*

$$\chi_p(G) \leqq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$$

Clearly $\chi_p(G) \geqq \left\lceil \frac{\Delta(G)}{2} \right\rceil$, so if $\Delta(G)$ is odd the conjecture amounts to saying that $\chi_p(G) = \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$, and in this case if true it would be a strengthening of Vizing's theorem [9, 10]. We have recently heard that the same conjecture is made in [1] and that the conjecture is proved for regular graphs of degree $\leqq 5$ in [1] and [8].

**Acknowledgement.** I would like to record with thanks helpful conversations with D. E. Daykin and R. Rado, and I would particularly like to thank L. D. Andersen who found several mistakes in draft versions of this paper.

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A. J. W. Hilton

*Department of Mathematics,*
*University of Reading*
*Reading, RG6 2AX, England*